13 PDEs on spatially bounded domains: initial boundary value problems (IBVPs)

A prototypical problem we will discuss in detail is the 1D diffusion equation

$u_t = Du_{xx}$	$0 < x < l, t > 0$	finite-length rod
$u(x, 0) = f(x)$	$0 < x < l$	initial heat or mass distribution

Now we have boundaries at $x = 0, l$, so we need to discuss *boundary conditions* (**b.c.s**) on the parabolic boundary (see figure 1). Using the language of u representing temperature, so $D = k/\rho c$, consider conditions at $x = 0$:

A. Prescribe the temperature at $x = 0$: $u(0, t) = h(t)$
This is a **Dirichlet** boundary condition, and often $h(t)$ is a constant. If we translate u so that $u(0, t) = 0$ in this case, then we have a **homogeneous Dirichlet b.c.**

B. Prescribe the flux at $x = 0$: $\frac{\partial u}{\partial x}(0, t) = h(t)$
This is a **Neumann** boundary condition. With the rate of heat transfer being $-k\frac{\partial u}{\partial x}$ (remember, k is thermal conductivity), we could write $-k\frac{\partial u}{\partial x}(0, t) = h$. The **homogeneous Neumann b.c.** is $\frac{\partial u}{\partial x}(0, t) = 0$, and is often interpreted as the (perfectly) *insulating* b.c. (no heat flow through the boundary, or no current through a conductor end).

C. Combination these conditions at $x = 0$: $au(0, t) + b\frac{\partial u}{\partial x}(0, t) = h(t)$
This is the **Robin** boundary condition, or **boundary condition of the third kind**. For example, if we think of Newton cooling at $x = 0$, we could consider a model of *imperfect* insulating condition in the thermal energy context. The constitutive law would be that the rate of heat loss flux density is proportional to the difference in temperature of the material and its surroundings. So, the flux is $J(0, t) = \pm h(u(0, t) - u_s(t))$, where h is a (constant) heat transfer coefficient, and u_s is the temperature of the surroundings. The value of h depends on the material, the type of material surroundings, the velocity of fluid flow (the "wind-chill" factor), etc. The particular physical problem being modeled will determine the correct sign to use on the right-hand side.Using Fourier's law, we would have $-k\partial u(0, t)/\partial x = \pm h(u(0, t) - u_s)$.

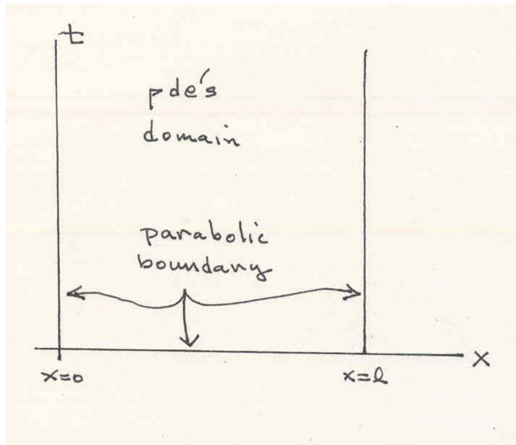

Figure 1: Domain of a 1D IBVP like (1), indicating the parabolic boundary.

A rescaling than gives $au(0,t) + b\frac{\partial u}{\partial x}(0,t) = h(t)$. Note that if $a \to 0$, we recover the Neumann b.c., while if $b \to 0$, then we recover the Dirichlet b.c. Another way of viewing the Robin b.c. is that it typifies physical situations where the boundary absorbs some, but not all, of the energy, heat, etc. being transmitted through it.

D. Continuity boundary conditions:
Suppose we have two materials bonded tightly together at $x = 0$. Then we might impose continuity of the temperature and its gradient across the boundary. That is, we would consider the boundary condition being both $u(0-,t) = u(0+,t)$ and $-k_1 u_x(0-,t) = -k_2 u_x(0+,t)$.

There are other possible boundary conditions. For example, the radiation boundary condition, which is a *nonlinear* boundary condition, has the form, in physical variables, of $-ku_x(0,t) = \epsilon\sigma\{u^4(0,t) - u_s^4\}$. We will not consider this boundary condition in these Notes.

Consider the parabolic (heat) problem

$$\begin{cases} u_t = Du_{xx} & 0 < x < l \ , \ t > 0 \\ u(x,0) = f(x) & 0 < x < l \\ u(0,t) = u_0(t) \ , \ u(l,t) = u_1(t) & t > 0 \end{cases} \tag{1}$$

Our goal is to obtain an explicit solution to this problem, with various boundary conditions, along with solving the analogous wave equation problems. Solutions will be in terms of **eigenfunction series**, so we need to learn the method of separation of variables, about eigenfunction problems, and about Fourier series. First we are going to discuss some properties associated with the heat equation on (spatially) finite domains.

Uniqueness: First we have to have the correct setting. Let $\Omega = \{(x,t) : 0 < x < l \ , \ t > 0\}$, then the closure of Ω is $\bar{\Omega} = \{(x,t) : 0 \le x \le l, t \ge 0\}$. Define the set of functions $C^{2,1}(\Omega)$ as the set of functions $w(x,t)$ defined on $\bar{\Omega}$, which are continuous on $\bar{\Omega}$, and w_t, w_x, w_{xx} are continuous in Ω. (That is, we do not assume derivatives are continuous on the boundary of Ω.)

Theorem: Problem (1) has at most one solution in $C^{2,1}(\Omega)$.

Proof: Suppose u_1, u_2 are two solutions in $C^{2,1}(\Omega)$. Let $v := u_1 - u_2$; then v is the solution to $v_t = Dv_{xx}$, $0 < x < l$, $t > 0$, and $v(0,t) = 0 = v(l,t)$ for $t > 0$. Let $V(t) := \frac{1}{2}\int_0^l (v(x,t))^2 dx$. Note that this function satisfies $V(0) = 0$ ($v(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0$), and $V(t) \ge 0$ for all $t \ge 0$. Also

$$dV/dt = \int_0^l vv_t \ dx = D\int_0^l vv_{xx} \ dx = D\{vv_x|_0^l - \int_0^l (v_x)^2 dx\}$$

$$= -D\int_0^l (v_x)^2 dx \le 0 \ \ ,$$

which implies $V(t) \equiv 0$ for all t; so $v(x,t) \equiv 0$ in $\bar{\Omega} \to u_1 \equiv u_2$ in $\bar{\Omega}$.

Remark: Notice in the proof that problem (1) can have Dirichlet or Neumann boundary conditions at either end for the boundary terms in the integration-by-parts expression to drop out. With only slightly more work, uniqueness result follows for Robin b.c.s also.

Exercises:

1. Reconsider problem (1) again, but with the boundary condition at $x = l$ replaced by the Robin condition $u_x(l,t) + au(l,t) = u_1(t)$, with $a > 0$. Repeat the above proof for uniqueness of solution of a solution to this new problem. Can a problem arise is $a < 0$? If so, does this say anything about the parameter value from a physical standpoint?

2. Now switch the Robin condition to the left boundary, that is, reconsider problem (1) with the left boundary condition replaced by $u_x(0,t) + au(0,t) = u_0(t)$. What must the sign of a be to guarantee uniqueness for the Robin-Dirichlet problem?

Theorem: Consider the problem

$$\begin{cases} u_t = Du_{xx} & 0 < x < l\ ,\ t > 0 \\ u(x,0) = f(x) & 0 < x < l \\ u_x(0,t) - au(0,t) = u_0(t)\ ,\ u_x(l,t) + bu(l,t) = u_1(t) & t > 0 \end{cases}$$

where $a, b > 0$ are constants. Then this problem has at most one solution in $C^{2,1}(0,l)$.

Exercise: Prove as in the theorem for Problem (1), but now you will obtain $DV/dt \leq -D\int_0^l (v_x)^2 dx - bv(l,t)^2 - av(0,t)^2 \leq 0$. Hence, from a physical standpoint, you must pay attention to having the correct signs on coefficients when dealing with boundary conditions.

Remark: It is very common in the study of pdes to use functionals like V to prove qualitative statements about solutions to pdes. The argument above is easily generalized to multidimensional domains and to more complicated equations. We will again see such "energy arguments" again later in the course.

Maximum Principle for Diffusion Equations

Theorem: If, for any $T > 0$, $u(x,t)$ is continuous on $\bar{\Omega}_T = \{(x,y) : 0 \leq x \leq l,\ 0 \leq t \leq T\}$ and the satisfies the diffusion equation in $\Omega_T = \{(x,y) : 0 < x < l,\ 0 < t < T\}$, then the maximum value of $u(x,t)$ is either on the boundary $t = 0$, or on the lateral sides $x = 0$ or $x = l$; that is, the maximum of $u(x,t)$ in Ω is less than or equal to the maximum of $u(x,t)$ on Ω's parabolic boundary.

Remark: The minimum value of $u(x,t)$ also is on the parabolic boundary because one can apply the maximum principle to $-u(x,t)$. There is a stronger version of the maximum principle that asserts that the maximum cannot be assumed anywhere inside the rectangle, but only on the parabolic boundary, unless u is a constant.

Well-posedness
Recall that a problem is well-posed if

1. It has a solution;
2. The solution is unique;
3. The solution is stable to small perturbations in boundary data.

We have just covered property #2 for a problem like (1), and most of what we do from here on in the course is concerned with property #1, so we now discuss property #3 in a couple of ways.

Stability "in the square-integral sense"
Again consider two solutions u_1 and u_2 to the equation in (1), and for simplicity and illustration here assume they agree on $x = 0$ and on $x = l$, but at $t = 0$, $u_1(x,0) = g_1(x)$, $u_2(x,0) = g_2(x)$. Remember, our goal here is to give you a sense of what *stability* means in this context without overwhelming you with generality. Again let $v \equiv u_1 - u_2$ and $V(t) = \frac{1}{2}\int_0^l (v(x,t))^2\, dx$. As we showed in the uniqueness argument, V is nonincreasing ($V(t) \leq V(0)$), so

$$\int_0^l (v(x,t))^2 dx \leq \int_0^l (v(x,0))^2 dx \rightarrow$$

$$\int_0^l (u_1(x,t) - u_2(x,t))^2 dx \leq \int_0^l (g_1(x) - g_2(x))^2 dx \quad .$$

The right-hand side of this inequality measures relative error between the initial data for the two solutions, while the left-hand side measures the difference in solutions at a later time. Hence, in this square-integral sense, two solutions that "start near each other" stay near each other.

Stability "in uniform sense"
The maximum/minimum principle also shows stability, but with a different way for measuring it.

Consider two solutions u_1 and u_2 to problem (1) with different initial conditions as above. Let $v = u_1 - u_2$; then $v(x,t) \le max_{\bar{\Omega}}|g_1(x) - g_2(x)|$. By the minimum principle, $v(x,t) \ge -max_{\bar{\Omega}}|g_1(x) - g_2(x)|$. That is, $max_{\bar{\Omega}}|u_1(x,t) - u_2(x,t)| \le max_{0 \le x \le l}|g_1(x) - g_2(x)|$, for all $t > 0$.

Now return to problem (1) and consider some special cases before tackling the general problem.

Example 1: u_0, u_1 are constants
In this case we look for a **steady state solution**, which can also be called an equilibrium solution or time-independent solution. Thus, let $u(x,t) = U(x)$. The importance in such a solution is the hope that $u(x,t) \to U(x)$ as $t \to \infty$, and so U takes on the dominant behavior of the solution as time increases. Put another way, if we write $u(x,t) = U(x) + v(x,t)$, then v becomes the *transient* part of the solution, and $v(x,t) \to 0$ as $t \to \infty$ for all x, which we show later. U being the steady state means it satisfies

$$0 = \frac{d^2U}{dx^2} \quad 0 < x < l \ , \quad U(0) = u_0, \ \ U(l) = u_1 \ .$$

This trivial ODE gives $U(x) = Ax + B$, and substituting in the boundary conditions gives $U(x) = (\frac{u_1 - u_0}{l})x + u_0$. Substituting $u(x,t) = U(x) + v(x,t)$ into (1) gives

$$u_t = 0 + v_t = D(U'' + v_{xx}) = Dv_{xx} \ , \quad \text{that is}$$

$$v_t = Dv_{xx} \ , \ \ 0 < x < l \ , \quad v(0,t) = 0 = v(l,t), \quad \text{for all } t > 0,$$

$$\text{with } \ v(x,0) = f^*(x) := f(x) - U(x) \ .$$

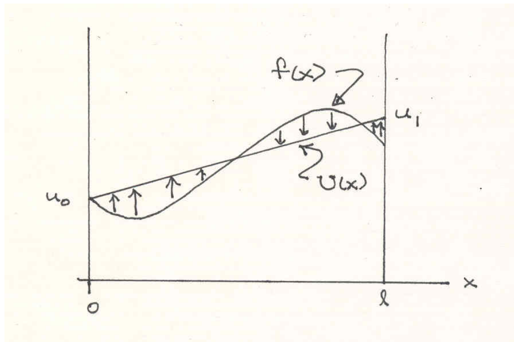

Figure 2: Expected behavior of Example 1 as $t \to \infty$.

Comment: We will see later that as $t \to \infty$, $v \to 0$ (uniformly in x). This convergence is exponentially fast, so after a short time, u "forgets" about the initial distribution f, and essentially all solution information is in the steady state solution. Figure 2 depicts this behavior. This is rather difficult to appreciate in viewing the above relatively elementary problem, but in more complicated modeling situations, the questions you might want to know might be answerable in just examining steady state behavior. For example, if instead we are faced with the three-dimensional problem

$$u_t = \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \quad \text{for } (x,y,z) \in \Omega \subset \mathbb{R}^3 \text{ , } t > 0$$

$$u(x,y,z,t) = h(x,y,z) \quad \text{for } (x,y,z) \in \partial\Omega = \text{boundary of } \Omega$$

$$u(x,y,z,0) = f(x,y,z) \quad \text{for } (x,y,z) \in \Omega$$

the same comment holds and thus, after a short time, u converges to the Dirichlet problem for Laplace's equation,

$$\begin{aligned} &\nabla^2 U = 0 \quad \text{in } \Omega \\ &U(x,y,z) = h(x,y,z) \quad \text{for } (x,y,z) \in \partial\Omega \end{aligned}$$

which is generally easier to solve.

Remark: In Figure 2 we have indicated that $f(0) = u_0$ (but $f(l) < u_1$). In general, if one has, say for problem (1), $\lim_{t\to 0} u_0(t) = \lim_{x\to 0} f(x)$, and $\lim_{t\to 0} u_1(t) = \lim_{x\to l} f(x)$, these are called **compatibility conditions**, and if one has them, the boundary condition all around the parabolic boundary is continuous. For the heat equation, and parabolic equations in general, non-continuous boundary data does not cause much problem because of the following

Smoothness Theorem: Let $u(x,t)$ be a solution of $u_t = Du_{xx}$ for $x \in (0,l)$, $t \in (0,T)$, for any $T > 0$. Then, $u(x,t)$ is infinitely differentiable with respect to both $x \in (0,l)$ and $t \in (0,T)$, and for each fixed t, u is an analytic function of x in its domain.

However, if the wave equation replaces the heat equation in this discussion, then, as we indicated with our discussion of the wave equation on the semi-infinite domain, a discontinuity in boundary data will be propagated along characteristics and is reflected back and forth between boundaries.

To solve problem (1) via separation of variables method, we *must* make such a transformation to homogeneous boundary conditions, or else the methodology does not work. To emphasize, in our strategy to solve our problem, the first step in the solution process is to transform the problem to one with homogeneous b.c.s if it does not already have them.

Example 2: $u_0 = u_0(t), u_1 = u_1(t)$ in (1); unless $u_0(t), u_1(t)$ converge to constants as $t \to \infty$, we do not have a steady state solution because of the t dependence, but we still have to abide by our previous comment about transforming to homogeneous boundary conditions. So write

$$u(x,t) = p(x,t) + v(x,t)$$

and choose **any** continuously differentiable function p that satisfies the boundary conditions. (For our specific case this means $p(0,t) = u_0(t), p(l,t) = u_1(t)$. One can see that $p(x,t) = (u_1(t) - u_0(t))(x/l) + u_0(t)$ works). Then $v(x,t)$ must be a solution to the problem

$$\begin{aligned} &v_t = Dv_{xx} + F(x,t) \qquad 0 < x < l,\ t > 0 \\ &v(x,0) = f^*(x) := f(x) - p(x,0) \qquad 0 < x < l \\ &v(0,t) = v(l,t) = 0 \qquad t > 0 \end{aligned} \tag{2}$$

where $F(x,t) = Dp_{xx}(x,t) - p_t(x,t)$. In our special example case, $F = -p_t = -\frac{du_0}{dt} - \frac{x}{l}(\frac{du_1}{dt} - \frac{du_0}{dt})$, but I am also presenting a general methodology. So we have converted a problem for u that has a homogeneous equation, but non-homogeneous b.c.s into a problem for v that has a non-homogeneous equation, but homogeneous b.c.s. That is ok because our solving technique can deal directly with the v problem (2).

Remark: Whether we have Dirichlet or Neumann b.c.s on either end of the interval domain, p being at most quadratic in x will work. For example, if our boundary conditions are $u_x(0,t) = u_0(t)$, $u(l,t) = u_1(t)$, then we can use $p(x,t) = (u_1(t) - lu_0(t))(x/l)^2 + u_0(t)x$. We can also use $p(x,t) = u_0(t)(x-l) + u_1(t)$.

Example 3: Neumann boundary conditions

$$u_t = Du_{xx} \qquad 0 < x < l,\ t > 0$$

$$u(x,0) = f(x) \qquad 0 < x < l$$

$$u_x(0,t) = 0 = u_x(l,t) \qquad t > 0$$

Now we have changed our boundary conditions from the previous example, so again seek a steady state solution $u(x,t) = U(x)$. Then upon substituting into the equation and boundary conditions, we have U satisfying

$$0 = \frac{d^2U}{dx^2}\ ,\ 0 < x < l$$

$$\frac{dU}{dx}(0) = 0 = \frac{dU}{dx}(l)$$

Hence, $U(x) = Ax + B$, but $dU/dx = A = 0$, so $U(x) \equiv B$, and arbitrary constant. But does this mean that there is an infinite number of solutions? We will find out later through Fourier series techniques that this is not the case, but we can *informally* determine what B should be without resorting to such techniques. Let us view the problem as a model again. We have a thin rod, insulated on each end, and on the lateral sides. The equation is homogeneous, we have no heat sources (or sinks), so the only dynamics is to redistribute the initial heat distribution according to Fourier's law. If we wait long long enough, we expect the temperature through the rod to be uniform.

That is consistent with $U(x) \equiv B$. Since we neither add or subtract heat energy, we have to live with what we have initially, so the final temperature should be an average of what we start with. Another way to look at this is to let $E(t) = \int_0^l u(x,t)dx$, and think of E as proportional to the total energy at time t. Then

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^l u(x,t)dx = \int_0^l \frac{\partial u}{\partial t}(x,t)dx = D\int_0^l \frac{\partial^2 u}{\partial x^2}(x,t)dx$$

$$= D\frac{\partial u}{\partial x}(x,t)|_0^l = 0$$

so $E(t)$ = constant with respect to $t > 0$. (This is a formal calculation because we have not justified interchanging differentiation and integration, but it turns out to be ok because of the smoothness of the solution u.) So, again informally,

$$E = \lim_{t\to\infty}\int_0^l u(x,t)dx = \int_0^l \lim_{t\to\infty} u(x,t)dx = \int_0^l U(x)dx = Bl$$

while

$$E = \lim_{t\to 0}\int_0^l u(x,t)dx = \int_0^l \lim_{t\to 0} u(x,t)dx = \int_0^l f(x)dx \quad .$$

Therefore, $B = \frac{1}{l}\int_0^l f(x)dx$ = average of $f(x)$ on $(0,l)$. Thus, $u(x,t) = (1/l)\int_0^l f(y)dy$ + transient part of the solution.

Summary: Know the different boundary conditions. Know the maximum principle and smoothness result for heat equation solutions. Since, to solve IBVPs we will need problems with homogeneous boundary conditions, you need to know how to transform a problem with non-homogeneous b.c.s to one with homogeneous b.c.s.

Exercises

For the following problems 1-4 with non-homogeneous boundary conditions, write $u(x,t) = p(x,t) + v(x,t)$ and determine a $p(x,t)$ that satisfies the boundary conditions, then write out the problem $v(x,t)$ satisfies if u satisfies the given problem.

1.
$$\begin{cases} u_t = Du_{xx} & 0 < x < l\ ,\ t > 0 \\ u(x,0) = f(x) & 0 < x < l \\ u_x(0,t) = 1\ ,\quad u(l,t) = 2 & t > 0 \end{cases}$$

2.
$$\begin{cases} u_t = u_{xx} & 0 < x < 1\ ,\ t > 0 \\ u(x,0) = f(x) & 0 < x < 1 \\ u(0,t) = 1\ ,\quad u_x(1,t) = 2e^{-t} & t > 0 \end{cases}$$

3.
$$\begin{cases} u_t = Du_{xx} & 0 < x < l\ ,\ t > 0 \\ u(x,0) = -1 & 0 < x < l \\ u_x(0,t) = u(0,t) - 1\ ,\quad u_x(l,t) = 0 & t > 0 \end{cases}$$

4.
$$\begin{cases} u_t = 4.3u_{xx} & 0 < x < 2\ ,\ t > 0 \\ u(x,0) = 0 & 0 < x < 2 \\ u_x(0,t) = u_0(t)\ ,\quad u(2,t) = u_1(t) & t > 0 \end{cases}$$

5. In the theory of combustion and explosions, the heat energy release T in a circular cylinder of radius $r = a$ and held at temperature T_0 at $r = a$ satisfies, under some restrictive assumptions, a model for $u = T - T_0$ of the form
$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + qe^{pu}\ ,$$
where p, k, and q are positive constants. This is a rather unpleasant **nonlinear** pde because of the e^{pu} term. If we first look for a steady state solution, $u = u(r)$ only, the model with boundary conditions becomes
$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + qe^{pu} = 0 \qquad 0 < r < a$$
$$u(a) = 0\ ,\ \frac{du}{dr}(0) = 0 \quad \text{radial symmetry condition}$$

We want to solve this nonlinear ODE up to an integration constant. (Because it is a nonlinear equation we should not expect a unique solution.)

(a) First let $v = r\frac{du}{dr}$ and obtain the equation for v, that is,

$$\frac{d}{dr}(\frac{1}{r}\frac{dv}{dr}) = \frac{p}{2r^2}\frac{d}{dr}(v^2) \ .$$

Hence,

$$\frac{p}{2}\frac{d}{dr}v^2 = r^2\frac{d}{dr}(\frac{1}{r}\frac{dv}{dr}) \tag{3}$$

(b) Use the fact that $r^2\frac{d}{dr}(\frac{1}{r}\frac{dv}{dr}) = \frac{d}{dr}(r\frac{dv}{dr} - 2v)$, and substitute this into (3), and integrate the equation. The boundary condition $v(0) = 0$ forces the constant of integration to be 0.

(c) Show that you can separate variables on this new equation to obtain $(\frac{1}{v} - \frac{p}{pv+4})dv = 2\frac{dr}{r}$. Upon integrating, obtain

$$v = \frac{4Cr^2}{1 - pCr^2}.$$

(d) u should be decreasing from $r = 0$ to $r = a$; that is, $\frac{du}{dr} < 0$, which will imply $v < 0$. For convenience, write $C = -A < 0$, so that $v = r\frac{du}{dr} = -\frac{4Ar^2}{1+pAr^2}$. Now integrate this equation again and use the boundary condition $u(a) = 0$ to obtain

$$u(r) = \frac{2}{p}\ln(\frac{1 + pAa^2}{1 + pAr^2}).$$

So the steady state temperature has the form $T(r) = -\frac{2}{p}\ln(1 + pAr^2) + T_0 + \frac{2}{p}\ln(1 + pAa^2)$.